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On the Smarandache-Pascal derived sequences and some of their conjectures

Xiaoxue Li and Di Han*

*Correspondence:
handi515@163.com
Department of Mathematics,
Northwest University, Xi'an, Shaanxi,
P.R. China

Abstract

For any sequence $\{b_n\}$, the Smarandache-Pascal derived sequence $\{T_n\}$ of $\{b_n\}$ is defined as $T_1 = b_1$, $T_2 = b_1 + b_2$, $T_3 = b_1 + 2b_2 + b_3$, generally, $T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot b_{k+1}$ for all $n \geq 2$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the combination number. In reference (Murthy and Ashbacher in Generalized Partitions and New Ideas on Number Theory and Smarandache Sequences, 2005), authors proposed a series of conjectures related to Fibonacci numbers and its Smarandache-Pascal derived sequence, one of them is that if $\{b_n\} = \{F_1, F_9, F_{17}, \ldots\}$, then we have the recurrence formula $T_{n+1} = 49 \cdot (T_n - T_{n-1})$, $n \geq 2$. The main purpose of this paper is using the elementary method and the properties of the second-order linear recurrence sequence to study these problems and to prove a generalized conclusion.

Keywords: Smarandache-Pascal derived sequence; Fibonacci number; combination number; elementary method; conjecture

1 Introduction

For any sequence $\{b_n\}$, we define a new sequence $\{T_n\}$ through the following method: $T_1 = b_1$, $T_2 = b_1 + b_2$, $T_3 = b_1 + 2b_2 + b_3$, generally, $T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot b_{k+1}$ for all $n \geq 2$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the combination number. This sequence is called the Smarandache-Pascal derived sequence of $\{b_n\}$. It was introduced by professor Smarandache in [1] and studied by some authors. For example, Murthy and Ashbacher [2] proposed a series of conjectures related to Fibonacci numbers and its Smarandache-Pascal derived sequence; three of them are as follows.

Conjecture 1 *Let $\{b_n\} = \{F_{8n+1}\} = \{F_1, F_9, F_{17}, F_{25}, \ldots\}$, $\{T_n\}$ be the Smarandache-Pascal derived sequence of $\{b_n\}$, then we have the recurrence formula*

$$T_{n+1} = 49 \cdot (T_n - T_{n-1}), \quad n \geq 2.$$

Conjecture 2 *Let $\{b_n\} = \{F_{10n+1}\} = \{F_1, F_{11}, F_{21}, F_{31}, \ldots\}$, $\{T_n\}$ be the Smarandache-Pascal derived sequence of $\{b_n\}$, then we have the recurrence formula*

$$T_{n+1} = 125 \cdot (T_n - T_{n-1}), \quad n \geq 2.$$

Conjecture 3 *Let $\{b_n\} = \{F_{12n+1}\} = \{F_1, F_{13}, F_{25}, F_{37}, \ldots\}$, $\{T_n\}$ be the Smarandache-Pascal derived sequence of $\{b_n\}$, then we have the recurrence formula*

$$T_{n+1} = 324 \cdot (T_n - T_{n-1}), \quad n \geq 2.$$

Regarding these conjectures, it seems that no one has studied them yet; at least, we have not seen any related results before. These conjectures are interesting; they reveal the profound properties of the Fibonacci numbers. The main purpose of this paper is using the elementary method and the properties of the second-order linear recurrence sequence to study these problems and to prove a generalized conclusion. That is, we shall prove the following.

Theorem *Let $\{X_n\}$ be a second-order linear recurrence sequence with $X_0 = u$, $X_1 = v$, $X_{n+1} = aX_n + bX_{n-1}$ for all $n \geq 1$, where $a^2 + 4b > 0$. For any positive integer $d \geq 2$, we define the Smarandache-Pascal derived sequence of $\{X_{dn+1}\}$ as*

$$T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot X_{dk+1}.$$

Then we have the recurrence formula

$$T_{n+1} = (2 + A_d + b \cdot A_{d-2}) \cdot T_n - \left(1 + A_d + b \cdot A_{d-2} + (-b)^d\right) \cdot T_{n-1},$$

where the sequence $\{A_n\}$ is defined as $A_0 = 1$, $A_1 = a$, $A_{n+1} = a \cdot A_n + b \cdot A_{n-1}$ for all $n \geq 1$. In fact this time, the general term is

$$A_n = \frac{1}{\sqrt{a^2 + 4b}} \left[\left(\frac{a + \sqrt{a^2 + 4b}}{2} \right)^{n+1} - \left(\frac{a - \sqrt{a^2 + 4b}}{2} \right)^{n+1} \right].$$

Now we take $b = 1$, then from our theorem, we may immediately deduce the following three corollaries.

Corollary 1 *Let $\{X_n\}$ be a second-order linear recurrence sequence with $X_0 = u$, $X_1 = v$, $X_{n+1} = aX_n + X_{n-1}$ for all $n \geq 1$. For any even number $d \geq 2$, we have the recurrence formula*

$$T_{n+1} = (2 + A_d + A_{d-2}) \cdot (T_n - T_{n-1}), \quad n \geq 2.$$

Corollary 2 *Let $\{X_n\}$ be a second-order linear recurrence sequence with $X_0 = u$, $X_1 = v$, $X_{n+1} = aX_n + X_{n-1}$ for all $n \geq 1$. For any odd number $d \geq 2$, we have the recurrence formula*

$$T_{n+1} = (2 + A_d + A_{d-2}) \cdot T_n - (A_d + A_{d-2}) \cdot T_{n-1}, \quad n \geq 2,$$

where

$$A_n = A_n(a) = \frac{1}{\sqrt{a^2 + 4}} \left[\left(\frac{a + \sqrt{a^2 + 4}}{2} \right)^{n+1} - \left(\frac{a - \sqrt{a^2 + 4}}{2} \right)^{n+1} \right].$$

It is clear that $F_{n+1}(a) = A_n(a)$ is a polynomial of a; sometimes, it is called a Fibonacci polynomial, because $F_n(1) = F_n$ is Fibonacci number, see [3–5].

If we take $a = 1$, $X_0 = 0$, $X_1 = 1$ in Corollary 1, then $\{X_n\} = \{F_n\}$ is a Fibonacci sequence. Note that $A_n = F_{n+1}$, $2 + A_8 + A_6 = 2 + F_9 + F_7 = 2 + 34 + 13 = 49$, $2 + A_{10} + A_8 = 2 + F_{11} + F_9 = 2 + 89 + 34 = 125$, $2 + A_{12} + A_{10} = 2 + F_{13} + F_{11} = 2 + 233 + 89 = 324$; from Corollary 1, we may immediately deduce that the three conjectures above are true.

If we take $a = 2$, $X_0 = P_0 = 0$, $X_1 = P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$ for all $n \geq 1$, then P_n are the Pell numbers. From Corollary 1, we can also deduce the following.

Corollary 3 *Let P_n be the Pell number. Then for any positive integer d and*

$$T_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot P_{2dk+1},$$

we have the recurrence formula

$$T_{n+1} = (2 + P_{2d+1} + P_{2d-1}) \cdot (T_n - T_{n-1}), \quad n \geq 2.$$

On the other hand, from our theorem, we know that if $\{b_n\}$ is a second-order linear recurrence sequence, then its Smarandache-Pascal derived sequence $\{T_n\}$ is also a second-order linear recurrence sequence.

2 Proof of the theorem

To complete the proof of our theorem, we need the following.

Lemma *Let integers $m \geq 0$ and $n \geq 2$. If the sequence $\{X_n\}$ satisfying the recurrence relations $X_{n+2} = a \cdot X_{n+1} + b \cdot X_n$, $n \geq 0$, then we have the identity*

$$X_{m+n} = A_{n-1} \cdot X_{m+1} + b \cdot A_{n-2} \cdot X_m,$$

where A_n is defined as $A_0 = 1$, $A_1 = a$ and $A_{n+1} = a \cdot A_n + b \cdot A_{n-1}$ for all $n \geq 1$, or

$$A_n = \frac{1}{\sqrt{a^2 + 4b}} \left[\left(\frac{a + \sqrt{a^2 + 4b}}{2} \right)^{n+1} - \left(\frac{a - \sqrt{a^2 + 4b}}{2} \right)^{n+1} \right].$$

Proof Now we prove this lemma by mathematical induction. Note that the recurrence formula $X_{m+2} = a \cdot X_{m+1} + b \cdot X_m$, $A_1 = a$, $A_0 = 1$, $A_{n+1} = a \cdot A_n + b \cdot A_{n-1}$ for all $n \geq 1$. So $X_{m+2} = A_1 \cdot X_{m+1} + b \cdot A_0 \cdot X_m$. That is, the lemma holds for $n = 2$. Since $X_{m+3} = a \cdot X_{m+2} + b \cdot X_{m+1} = a \cdot (a \cdot X_{m+1} + b \cdot X_m) + b \cdot X_{m+1} = (a^2 + b) \cdot X_{m+1} + ba \cdot X_m = A_2 \cdot X_{m+1} + bA_1 \cdot X_m$. That is, the lemma holds for $n = 3$. Suppose that for all integers $2 \leq n \leq k$, we have $X_{m+n} = A_{n-1} \cdot X_{m+1} + b \cdot A_{n-2} \cdot X_m$. Then for $n = k + 1$, from the recurrence relations for X_m and the inductive hypothesis, we have

$$\begin{aligned} X_{m+k+1} &= a \cdot X_{m+k} + b \cdot X_{m+k-1} \\ &= a \cdot (A_{k-1} \cdot X_{m+1} + b \cdot A_{k-2} \cdot X_m) + b \cdot (A_{k-2} \cdot X_{m+1} + b \cdot A_{k-3} \cdot X_m) \\ &= (a \cdot A_{k-1} + b \cdot A_{k-2}) \cdot X_{m+1} + b \cdot (a \cdot A_{k-2} + b \cdot A_{k-3}) \cdot X_m \\ &= A_k \cdot X_{m+1} + b \cdot A_{k-1} \cdot X_{m-1}. \end{aligned}$$

That is, the lemma also holds for $n = k + 1$. This completes the proof of our lemma by mathematical induction. □

Now, we use this lemma to complete the proof of our theorem. For any positive integer d, from the definition of T_n and the properties of the binomial coefficient $\binom{n}{k}$, we have

$$\begin{aligned}\binom{n-1}{k}+\binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!}+\frac{(n-1)!}{(k-1)!(n-k)!}\\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{k}+\frac{1}{n-k}\right)=\binom{n}{k} \end{aligned}\tag{1}$$

and

$$\begin{aligned}T_{n+1} &= \sum_{k=0}^{n}\binom{n}{k}\cdot X_{dk+1}\\ &= X_1+X_{dn+1}+\sum_{k=1}^{n-1}\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right)X_{dk+1}\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}\cdot X_{dk+1}+\sum_{k=0}^{n-2}\binom{n-1}{k}\cdot X_{dk+d+1}+X_{dn+1}\\ &= T_n+\sum_{k=0}^{n-1}\binom{n-1}{k}X_{dk+d+1}. \end{aligned}\tag{2}$$

From the lemma, we have$X_{dk+d+1} = A_d \cdot X_{dk+1} + b \cdot A_{d-1} \cdot X_{dk}$, by (2) and the definition of T_n, we may deduce that

$$\begin{aligned}T_{n+1} &= T_n+\sum_{k=0}^{n-1}\binom{n-1}{k}\cdot(A_d\cdot X_{dk+1}+b\cdot A_{d-1}\cdot X_{dk})\\ &= T_n+A_d\cdot\sum_{k=0}^{n-1}\binom{n-1}{k}\cdot X_{dk+1}+b\cdot A_{d-1}\cdot\sum_{k=0}^{n-1}\binom{n-1}{k}\cdot X_{dk}\\ &= (1+A_d)\cdot T_n+b\cdot A_{d-1}\cdot\sum_{k=0}^{n-1}\binom{n-1}{k}\cdot X_{dk}. \end{aligned}\tag{3}$$

On the other hand, from the lemma, we also have $X_{dk+d} = A_{d-1} \cdot X_{dk+1} + b \cdot A_{d-2} \cdot X_{dk}$, from this and formula (1), we have

$$\begin{aligned}\sum_{k=0}^{n-1}\binom{n-1}{k}\cdot X_{dk} &= X_0+X_{d(n-1)}+\sum_{k=1}^{n-2}\binom{n-1}{k}\cdot X_{dk}\\ &= X_0+X_{d(n-1)}+\sum_{k=1}^{n-2}\left(\binom{n-2}{k}+\binom{n-2}{k-1}\right)\cdot X_{dk}\\ &= \sum_{k=0}^{n-2}\binom{n-2}{k}\cdot X_{dk}+\sum_{k=0}^{n-3}\binom{n-2}{k}\cdot X_{dk+d}+X_{d(n-1)}\\ &= \sum_{k=0}^{n-2}\binom{n-2}{k}\cdot X_{dk}+\sum_{k=0}^{n-2}\binom{n-2}{k}\cdot(A_{d-1}\cdot X_{dk+1}+b\cdot A_{d-2}\cdot X_{dk})\\ &= (1+b\cdot A_{d-2})\cdot\sum_{k=0}^{n-2}\binom{n-2}{k}\cdot X_{dk}+A_{d-1}\cdot T_{n-1}. \end{aligned}\tag{4}$$

From (3), we can also deduce that

$$T_n = (1 + A_d) \cdot T_{n-1} + b \cdot A_{d-1} \cdot \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk}. \tag{5}$$

Now, combining (3), (4) and (5), we may immediately get

$$\begin{aligned} T_{n+1} &= (1 + A_d) \cdot T_n + b \cdot A_{d-1} \cdot \left((1 + b \cdot A_{d-2}) \cdot \sum_{k=0}^{n-2} \binom{n-2}{k} \cdot X_{dk} + A_{d-1} \cdot T_{n-1} \right) \\ &= (1 + A_d) \cdot T_n + b \cdot A_{d-1}^2 \cdot T_{n-1} + (1 + b \cdot A_{d-2}) \cdot \big(T_n - (1 + A_d) \cdot T_{n-1} \big) \end{aligned}$$

or equivalent to

$$\begin{aligned} T_{n+1} &= (2 + A_d + b \cdot A_{d-2}) \cdot T_n - \big(1 + A_d + b \cdot A_{d-2} + b \cdot A_d \cdot A_{d-2} - b \cdot A_{d-1}^2 \big) \cdot T_{n-1} \\ &= (2 + A_d + b \cdot A_{d-2}) \cdot T_n - \big(1 + A_d + b \cdot A_{d-2} + (-b)^d \big) \cdot T_{n-1}, \end{aligned} \tag{6}$$

where we have used the identity

$$\begin{aligned} A_d \cdot A_{d-2} - A_{d-1}^2 &= \frac{-(-b)^{d-1}}{a^2 + 4b} \left[\left(\frac{a + \sqrt{a^2 + 4b}}{2} \right)^2 + \left(\frac{a - \sqrt{a^2 + 4b}}{2} \right)^2 \right] + \frac{2(-b)^d}{a^2 + 4b} \\ &= -(-b)^{d-1} \cdot \frac{a^2 + 2b + 2b}{a^2 + 4b} = -(-b)^{d-1}. \end{aligned}$$

Now, our theorem follows from formula (6).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XL studied the Smarandache-Pascal derived sequences and proved a generalized conclusion. DH participated in the research and summary of the study.

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